Resynchronization of sequential measurements using the Maximally-Coherent Reference technique

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Abstract

Remote sensing of a physical field generated by a small number of sources is limited by the size of the available array of sensors and by the array density. These limitations may lead to asynchronously measuring the field by sequentially moving a small prototype array around its facets, however at the cost of losing the phase between every array position. Re-synchronization using references, i.e. fixed sensors, can be used for phase retrieval, given that these references are of a number that is at least equal to the stochastic dimension of the field, and are not inter-correlated to the degree that hinder them from spanning the source signals' subspace. However, when the number of references largely exceeds the number of sources, the reference cross spectral matrix becomes ill-conditioned leading to the irrelevance of the least squares solution (LSS). Although the truncated singular value decomposition (TSVD) was successfully applied in the literature to solve this problem, its validity is restricted only to the case of scalar noise on the references. It is also very hard to set a threshold, for truncation, among the singular values when the references cross-spectral matrix is well-conditioned due to high noise levels. In this paper, a solution based on finding a set of virtual references that is maximally correlated with the field measurements, named the Maximally-Coherent Reference Technique (MCR), is proposed for re-synchronization. The method is validated by numerical simulations, and its results are compared to those of the LSS and the TSVD when employed for the same purpose.

1 Introduction

Inverse acoustic reconstruction, as an example of remote sensing of physical fields, is limited by the size of the used microphone array and by the microphone density, and they are inversely related, given a fixed number of microphones. In other words, a large number of microphones is needed to build a large array that envelopes the sound source, with a sufficient spatial resolution, what is practically limited by the cost and the available hardware (e.g. available number of channels in the acquisition system). This problem was practically solved by asynchronously measuring the sound field by sequentially moving a small prototype array around the source, however at the cost of losing the phase between each array position, hence, losing the cross-spectra between the sequential measurements which are needed for acoustic reconstruction. Hence, a re-synchronization technique was needed to recover the phase information. Fortunately, this re-synchronization is easily accomplished by the use of fixed sensors (references), given that these references are of a number that is at least equal to the stochastic dimension of the field, and are not inter-correlated to the degree that hinder them from spanning the source signals' subspace. In such a case, the full field cross-spectral matrix can be indirectly computed using both the references cross-spectral matrix and the cross-spectral matrix between the references and the array signals. The earliest attempt of the practical implementation of this technique was done by Hald [1], in the context of Near-field Acoustic Holography, and he coined his technique as "The Spatial Transformation of Sound Fields (STSF)".

If noise exists on the array microphones, and the target is to retrieve only the contribution of the sources without noise, the problem is called source separation/denoising [2]. Source separation can be done simultaneously with re-synchronization (i.e. while building the cross-spectral matrix of the entire field) by restricting the rank of the cross-spectral matrix of the field to that of the cross-spectral matrix of the references [3]. However if the reference signals themselves are noisy, the rank of their cross-spectral matrix is no longer equal to the number of the uncorrelated sources contributing to the field, and hence, a rank reduction strategy needs to be implemented on the cross-spectral matrix of the references. A strong challenge facing the rank reduction techniques arises when the noise levels on the references are very high to the degree that results in a well-conditioned references cross-spectral matrix (i.e. a full rank matrix with gradually decreasing singular values). In this situation, there is no clear threshold that divides some principal (important) auto-spectra from negligible noise auto-spectra, hence, the truncation process becomes very difficult.

Two main rank reduction techniques were proposed in the literature to tackle this problem: The Partial Coherence technique, presented by Bendat and Piersol [4], and the Virtual Coherence technique proposed by Price and Bernhard [5]. However the ordering operation of the reference signals required for the Partial Coherence technique is essentially problematic, as a model with arbitrary M inputs, will theoretically require analysing an $M!$ models which are conditioned by different input orders. To avoid this, one has to have a priory knowledge on the cause-and-effect relations between every pair of inputs, which is practically not guaranteed [4]. Besides, in the case of systems with large number of references (inputs), the probability of existences of perfectly correlated inputs (linearly dependent) arises, what would lead to numerical problems (singularity) during the conditioning operation [5, 6]. In addition to this, if the noises on the references are correlated, this technique is conceptually no longer valid, because it is based on the assumption of uncorrelated noises among channels.

While the Virtual Coherence technique, which is based on the principal component analysis (PCA), i.e. the truncated singular value decomposition (TSVD) of the matrix of references, has its validity restricted only to the case of scalar noise on the references (i.e. the covariance matrix of the noise is proportional to identity), otherwise, the obtained virtual spectra no longer represent the principal spectra of the sound field. Hence, it will fail in the cases of correlated noise on the references, and/or noises of different variances (e.g. reference sensors of different types) [7].

This paper adopts the Maximally-Coherent Reference (MCR) technique, recently introduced by the authors [7], for finding a set of virtual spectra, which are necessarily linear combinations of the measured references, so that they are maximally correlated with the array signals and so with the actual sources. This technique showed its immunity against the limitations of both the LSS and the TSVD, what makes it a robust basis for re-synchronization. The effectiveness of the method in the re-synchronization of sequential measurements is proved by numerical simulations.

2 Problem statement

Let $\mathbf{Y} \in \mathbb{C}^{M\times I}$ denote a set of complex Fourier coefficients of measurements returned by M output sensors and recorded for I independent snapshots, where $I \geq M$. The data $\mathbf{Y}$ are supposed to be produced by S sources, say $\mathbf{S} \in \mathbb{C}^{S\times I}$, whose contributions are noted $\mathbf{X} = \mathbf{HS} \in \mathbb{C}^{M\times I}$ for some linear but unknown operator $\mathbf{H} \in \mathbb{C}^{M\times S}$, and corrupted by additive disturbances $\mathbf{N} \in \mathbb{C}^{M\times I}$ uncorrelated with $\mathbf{X}$, such that

$$\mathbf{Y} = \mathbf{X} + \mathbf{N}. \tag{1}$$

It is assumed that the number of output sensors M exceeds or is equal to the number of sources, i.e. $M \geq S$. The data $\mathbf{Y}$ comes with a set of R references, $\mathbf{R} \in \mathbb{C}^{R\times I}$, which are supposed to be perfectly correlated with the S sources, in the sense that their exists a linear but unknown operator $\mathbf{L}$ such that

$$\mathbf{R} = \mathbf{LS}. \tag{2}$$

In practice, a small amount of additive noise may be present too in the references, such that

$$\mathbf{R} = \mathbf{LS} + \epsilon\mathbf{E}, \tag{3}$$

with $\epsilon \geq 0$ arbitrarily small, and where the noise covariance matrix on the references, $\mathbb{C}_{EE} = \mathbb{E}\{\mathbf{EE}^H\}$, is not necessarily proportional to the identity.

It is further assumed that the number of references exceeds or is equal to the number of sources, i.e. $R \geq S$. The aim is to predict the contributions $\mathbf{X}$ from the references $\mathbf{R}$, i.e. to find an operator $\mathbf{G} \in \mathbb{C}^{M\times R}$ such that

$$\hat{\mathbf{X}} = \mathbf{GR} \tag{4}$$

is an estimate of $\mathbf{X}$.

3 The proposed solution: the Maximally-Coherent Reference technique

A simple solution to the above problem that is often used in practice is provided by the least square estimate

$$\mathbf{G} = \operatorname{Arg}\min_{\mathbf{A}} \|\mathbf{Y} - \mathbf{A}\mathbf{R}\|^2 = \mathbf{S}_{YR}\mathbf{S}_{RR}^{-1}, \tag{5}$$

with $\mathbf{S}_{RR} = \mathbf{R}\mathbf{R}^H/I$ and $\mathbf{S}_{YR} = \mathbf{Y}\mathbf{R}^H/I$. Unfortunately, the matrix $\mathbf{S}_{RR}$ might be badly conditioned when $R > S$ and ϵ is small; it tends to be singular when $\epsilon \to 0$ and $R > S$.

A popular way to deal with the aforementioned situation is to replace the set of R references $\mathbf{R}$ by their first S principal components, as returned by TSVD. However, the so selected S components do not need to be the best ones to predict the contributions $\mathbf{X}$, i.e., the most correlated with the sources $\mathbf{S}$.

In light of this discussion, a better choice is to select S linear combinations of references $\mathbf{R}$ that best predict the data $\mathbf{Y}$. This amounts to finding a vector $\mathbf{b}_1$ such that $\mathbf{z}_1 = \mathbf{b}_1^H\mathbf{R}$ is maximally correlated with $\mathbf{Y}$, then to remove the contribution of $\mathbf{z}_1$ from the data and to find the next linear combination of references defined by vector $\mathbf{b}_2$ such that $\mathbf{z}_2 = \mathbf{b}_2^H\mathbf{R}$ is maximally correlated with the residual, etc. The proposed algorithm reads

- Set $\mathbf{Y}_0 = \mathbf{Y}$
- FOR $i = 1$ to S
  - Find $\mathbf{b}_i$: $\max \operatorname{corr}(\mathbf{Y}_i, \mathbf{b}_i^H\mathbf{R})$
  - Set $\mathbf{z}_i = \mathbf{b}_i^H\mathbf{R}$
  - Find $\mathbf{c}_i$: $\min_{\mathbf{c}} \|\mathbf{Y} - \mathbf{c}\mathbf{z}_i\|^2$
  - Set $\mathbf{Y}_i = \mathbf{Y}_{i-1} - \mathbf{c}_i\mathbf{z}_i$
- $i \leftarrow i + 1$
- END if $i > S$

Let denote $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_S]$ and $\mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_S]$ the values obtained from the above algorithm. Then, an estimate of the contribution $\mathbf{X}$ in the data $\mathbf{Y}$ is returned by

$$\hat{\mathbf{X}} = \sum_{s=1}^{S} \mathbf{c}_s\mathbf{b}_s^H\mathbf{R} = \mathbf{C}\mathbf{B}^H\mathbf{R}. \tag{6}$$

This algorithms is equivalent to the generalized eigenvalue decomposition (GEVD) of the pair of matrices $(\mathbf{S}_{RY}\mathbf{\Gamma}\mathbf{S}_{YR}, \mathbf{S}_{RR})$, where only the eigenvectors associated to the S largest eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_S \geq 0$ are conserved, for some positive definite matrix $\mathbf{\Gamma} \in \mathbb{C}^{M\times M}$, which can be chosen to be the identity matrix $\mathbf{I}$ or the output cross-spectral matrix $\mathbf{S}_{YY}^{-1}$ for instance. It is clear by looking at the first matrix of this generalized eigenvalue problem that, if $M < R$, then the rank of that first matrix is M, and there is no hope to search for more than M sources. This means that one must have $M \geq S$ if the effects of all sources are to be recovered, as initially assumed.

The matrix $\mathbf{B}$ then will be the matrix consisting of the eigenvectors as its columns, and $\mathbf{C} = \mathbf{S}_{YR}\mathbf{B}$. Equation (6) then becomes

$$\hat{\mathbf{X}} = \mathbf{C}\mathbf{B}^H\mathbf{R} = \mathbf{S}_{YR}\mathbf{B}\mathbf{B}^H\mathbf{R}. \tag{7}$$

As compared to Equation (5), the above result replaces the (possibly unstable or non-existing) inverse operator $\mathbf{S}_{RR}^{-1}$ by the projector $\mathbf{B}\mathbf{B}^H$.

4 Numerical validation

In order to verify the effectiveness of the MCR technique in solving the re-synchronization problem, numerical simulations were performed to compare the results of the MCR method to the results of both the LSS, specified in Equation (5), which uses the full $\mathbf{S}_{RR}$ matrix, and the TSVD, which decomposes the $\mathbf{S}_{RR}$ and then keep only the important eigenvalues (and their corresponding eigenvectors), then substitutes for $\mathbf{S}_{RR}$ in Equation

(5). For this experiment the number of the preserved general eigenvalues in the MCR method and the number of preserved eigenvalues in the TSVD both were equally set to the number of simulated sources (S). In the simulation, the number of sources was $S = 1$, the number of references was $R = 10$, the number of the output sensors was $M = 25$, and the number of snapshots was $I = 70$.

The sources were simulated as an iid complex Gaussian random processes with zero mean and unit variance, and were mapped to the output sensors and to the references using a complex-valued transfer matrix $\mathbf{H} \in \mathbb{C}^{(R+M)\times N}$, whose elements were randomly drawn from an iid complex Gaussian random processes with zero mean and unit variance. In order to give the simulation its spatial dimension, the output sensors were arranged in the form of a 5×5 array. To simulate a field emitted by a single point source (monopole), the sensor in the middle of the array (sensor no. 13) had its initially simulated power increased 5 times, while all the other sensors had their initially simulated powers multiplied by 0.02.

Uncorrelated noises on the output sensors were simulated as random complex Gaussian variables with zero mean, and their variances were specified so as to give output sensors' signal to noise ratio (SNR) of -10 dB ($\text{SNR}_{out} = -10$ dB) on each output signal. Noises on the references were simulated as random complex Gaussian variables with zero mean, and their variances were assigned so as to give reference sensors' SNR of 0 dB ($\text{SNR}_{ref} = 0$ dB) on each reference signal. Using the same SNR for all the sensors implies different noise power levels, which create a difficulty in the face of the TSVD. Also using such high noise levels on the references brings about another difficulty in the face of the TSVD. Finally, an 0.9 cross-correlation factor was imposed on the noises of the references, i.e. the references' noise covariance matrix is no more diagonal, what entails the failure of the TSVD.

Figure 1 shows the radiation maps of the true coherent spectra X, where X is a single snapshot of $\mathbf{X}$ (in the first row), the noisy spectra Y, where Y is a single snapshot of $\mathbf{Y}$ (in the second row), and the estimated coherent spectra using the MCR solution (MCRS), the TSVD solution (TSVDS), and the LSS (in the third, fourth and fifth row respectively). The results are shown in the forms of magnitude, phase, real, and imaginary parts, (in columns a,b,c and d respectively). It is clear from the second row that huge amplitude and phase distortion were caused by the addition of noise. The MCRS returned very good results, as one can see that the coherent spectra were well retrieved in terms of amplitude and phase. While the TSVDS recovered neither the amplitude nor the phase, in the contrary there was a huge underestimation of the coherent spectra. Finally, the LSS tend towards returning the noisy signal's magnitude, by definition, while its performance in the phase restoration was poor.

It is clear from this simulations that the MCRS is robust and immune against the situations that represents challenges for the LSS and the TSVDS, such as low SNR, correlated references with correlated noise, and/or with noise of heterogeneous powers (like signals coming from different types of sensors).

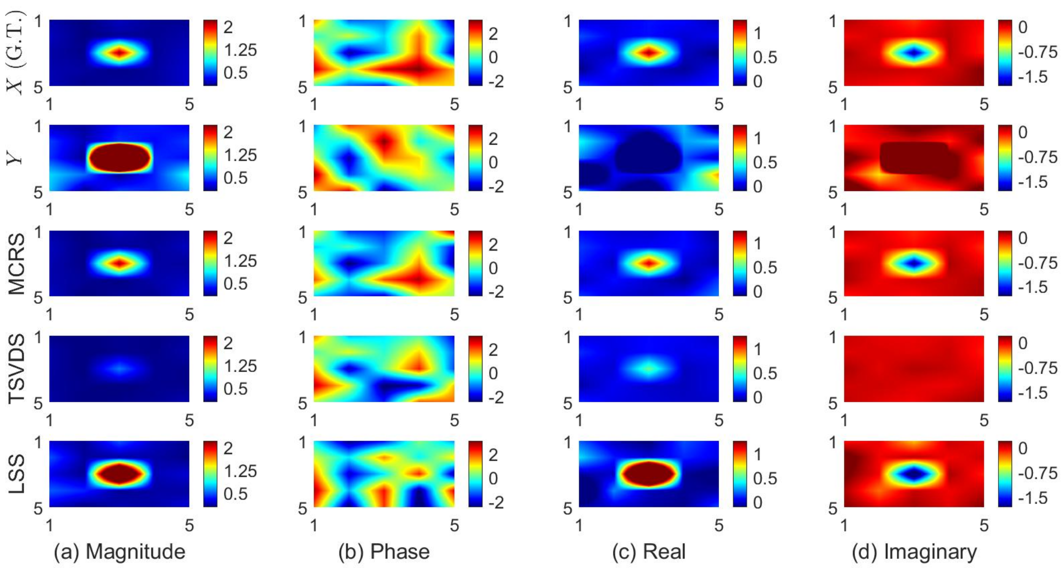

Figure 1: Results of the simulations: the radiation maps of the (a) magnitude, (b) phase, (c) real, and (d) imaginary parts of a single snapshot of the; first row: coherent spectra X (the ground truth G.T.), second row: noisy spectra Y, third row: MCRS, fourth row: TSVDS, and fifth row: LSS.

5 Conclusions

The challenges that appear while solving the source separation and/or the re-synchronization problems associated with the remote sensing of a coherent physical field (e.g. an acoustical field) are mainly emerging from the condition of the cross-spectral matrix of the references. While a badly conditioned cross-spectral matrix results in the failure of the LSS, a well conditioned cross-spectral matrix, due to high and different levels of noise, causes the TSVDS to fail. Hence, the motivation was to find a solution that mediates these two solutions, in the sense that it includes all the merits of the two, and excludes all the weaknesses of the two, as possible. The MCR technique was designed to search this (optimum) solution, and the objective of the optimization was to find the solution that is as close as possible to the coherent spectra. The solution was given through finding a set of virtual references, which are essentially linear combinations of the measured ones, that is maximally correlated with the field measurements and so with the true sources. This novel idea when tested in simulations, it returned excellent results compared to those of the LSS and the TSVDS, in terms of both source separation and re-synchronization.

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