Efficient preconditioners for the solution of a Regularized Digital Image Correlation (RDIC) problem

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Mathematical formulation of the DIC problem

The Digital Image Correlation [Bay et al, 1999] (DIC) problem is formalized as the non-linear optimization problem, namely the grey level conservation law,

$$\min_{\mathbf{u} \in L^2(\Omega)} \phi(\mathbf{u}) = \int_{\Omega} [f(x) - g(x + \mathbf{u}(x))]^2 dx. \quad (1)$$

with $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ the domain of interest, f, g the greyscale image of respectively the reference and the deformed specimen, and $u \in L^2(\Omega)$ the unknown displacement field.

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$$\mathrm{A}\delta u^{(k)} = b^{(k)} \quad \text{with} \quad \begin{cases} \mathrm{A}_{i,j} = \int_\Omega N_i^\mathsf{T} \nabla f(x) \nabla f(x)^\mathsf{T} N_j dx \\ \\ b_i^{(k)} = \int_\Omega N_i^\mathsf{T} \nabla f(f(x) - g(x + \mathbf{u}^{(k)}))dx \end{cases} . \qquad (2)$$

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Hence the need to solve a **sequence of linear systems involving a symmetric positive definite** A**, of order** $\#$**DOFs**.

Regularization of the problem

The problem (1) might need a regularization term to lower the measurement uncertainty,

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Different regularizations (Tikhonov, elastic) yield different matrices R, and finding the optimal value for α is not trivial.

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Method and features

Why using iterative methods rather than direct methods ?

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Let $\mathrm{A}, \mathrm{M} \in \mathbb{R}^{n \times n}$ be two s.p.d. linear operators, and $b, x_0 \in \mathbb{R}^n$. Let $r_0 = b - \mathrm{A}x_0$ denote the initial residual. The preconditioned conjugate gradient [Hestenes et al, 1952] yields after p iterations,

$$x_p = x_0 + \arg\min_{y \in \mathrm{K}_p} \|x^\star - y\|_{\mathrm{A}} \,,$$

with $x^\star = \mathrm{A}^{-1} b$ the exact solution, and Krylov subspace K_p,

$$\mathrm{K}_p = \operatorname{span}\left\{ M r_0, (\mathrm{MA})\mathrm{M} r_0, \ldots, (\mathrm{MA})^{p-1} \mathrm{M} r_0 \right\}.$$

Efficient preconditioner

The design of an efficient preconditioner [Wathen, 2015] is complex and mostly problem dependent. However, elementary algebraic preconditioners exist and can be easily implemented and tested.

An efficient preconditioner must ideally,

- Be cheap to construct and to store,
- Be cheap to apply to a vector ($M \approx A^{-1}$) or solve for a vector ($M \approx A$),
- Eventually be matrix-free,
- Allow parallelization.

Algebraic Preconditioners

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- $M = D^{-1}$: Jacobi or diagonal preconditioner,
- $M = (L^\top + D)^{-1} D (L + D)^{-1}$: Symmetric Gauss-Seidel preconditioner,
- Algebraic Multigrid used as preconditioners [Trottenberg et al, 2001],
- Domain Decomposition preconditioners [Dolean et al, 2015],
- Projection-based preconditioners: **Deflation** [Frank et al, 2001].

Deflation technique

Let $\mathcal{S} \subset \mathbb{R}^n$ and $S \in \mathbb{R}^{n \times k}$ such that $\mathcal{S} = \text{span}\{S\}$ be a block vector matrix and let us consider,

$$\pi_A(\mathcal{S}) = S(S^\top AS)^{-1}S^\top A.$$

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The solution of $\mathrm{A}x^\star = b$ can be written $x^\star = S(S^\mathsf{T} A S)^{-1} S b + \widetilde{x}$ with,

$$(\mathrm{I}_n - \pi_A(\mathcal{S}))^\mathsf{T} A \widetilde{x} = (\mathrm{I}_n - \pi_A(\mathcal{S}))^\mathsf{T} b. \tag{3}$$

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The so-called deflated linear system is such that,

- The deflated operator is symmetric positive semi-definite and the system is consistent,
- The operator null space is $\mathcal{S}$,
- $\kappa((\mathrm{I}_n - \pi_A(\mathcal{S}))^\mathsf{T} A) \leq \kappa(A)$.

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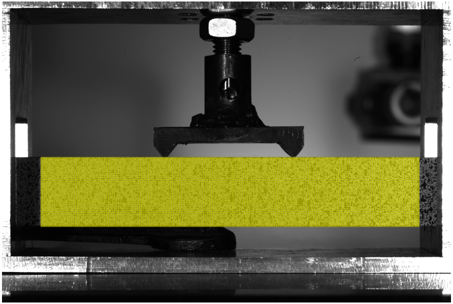
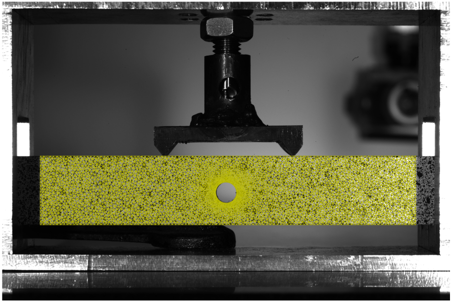
Investigated problems

Two different cases are studied here:



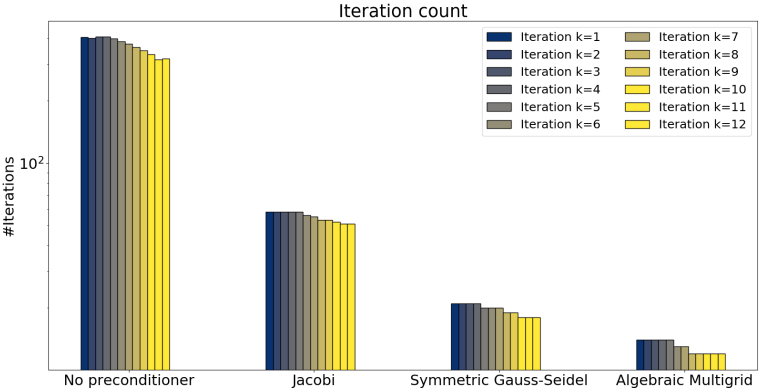
(a) High resolution mesh $n \approx 10^5$.



(b) Mesh with a hole $n \approx 10^4$.

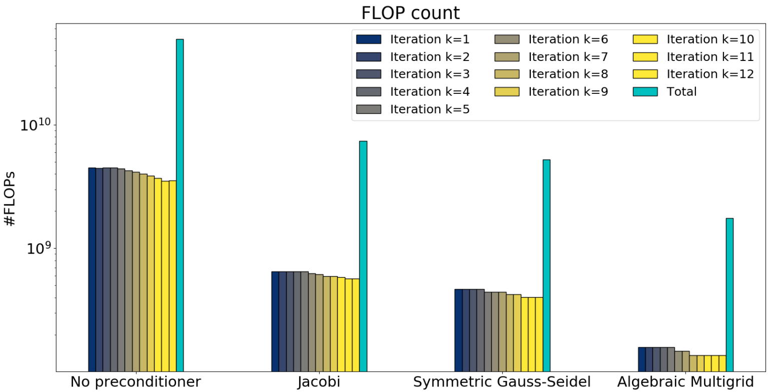
For both: Elastic regularization with $\alpha = 5 \cdot 10^3$

Iteration count vs. Preconditioners

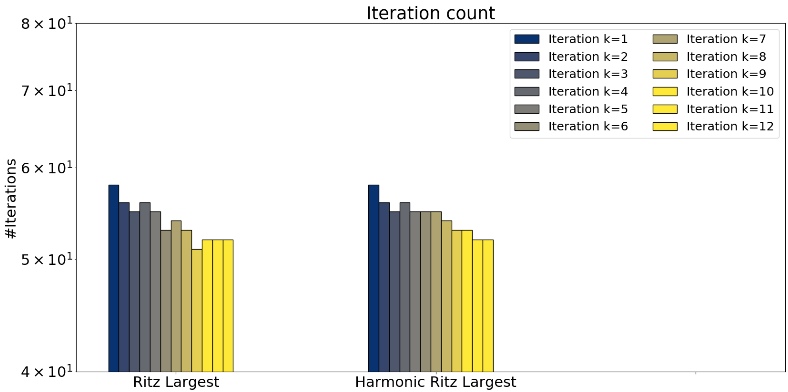


Algebraic Multigrid implementation, PyAMG: https://github.com/pyamg/pyamg

FLOP count vs. Preconditioners



Iteration count vs. Deflation strategy



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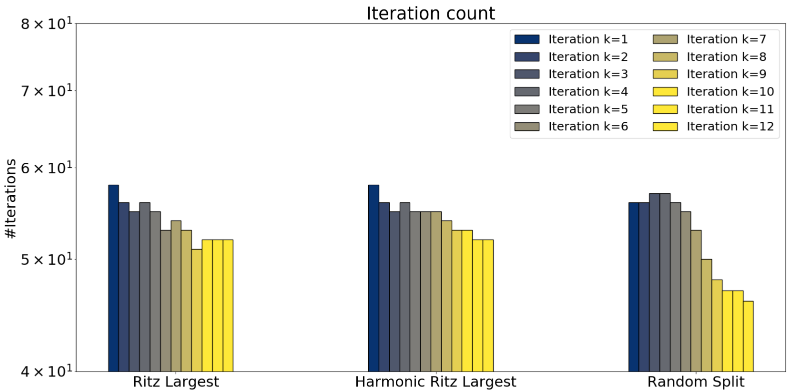

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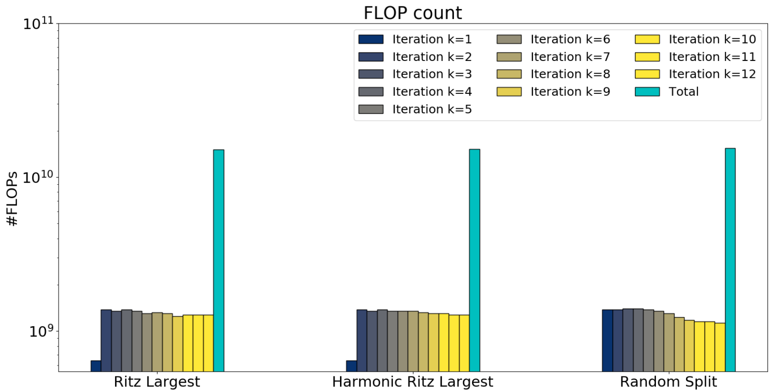

Conclusions and perspectives

Conclusions,

- Algebraic preconditioners perform well, especially the algebraic multi-grid,
- Random deflation subspace is potentially interesting compared to standard deterministic deflation strategies.

Perspectives,

- Interested in solving larger problems, with more complex meshes,
- Studying the performance of the preconditioners in this context,
- Combining preconditioning and deflation,
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Thank you for your attention.

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